Example of a Super-Ring without Super DP Structure.

I will begin with an example over $\mathbb{F}_3$ and then generalize to a whole class of examples over $\mathbb{F}_p$.

The first example is will be a quotient ring of $\mathbb{F}_3[\xi_1, \ldots, \xi_{24}]$. For the sake of brevity I will define some symbols.

$$\begin{aligned} a &= \xi_1\xi_2 + \xi_3\xi_4 \\ b &= \xi_5\xi_6 + \xi_7\xi_8 \\ c &= \xi_9\xi_{10} + \xi_{11}\xi_{12} \\ d &= \xi_{13}\xi_{14} + \xi_{15}\xi_{16} \\ e &= \xi_{17}\xi_{18} + \xi_{19}\xi_{20} \\ f &= \xi_{21}\xi_{22} + \xi_{23}\xi_{24} \end{aligned}$$

Notice that $a^2 = 2\xi_1\xi_2\xi_3\xi_4 \neq 0$ and $a^3 = 0$. This is true for each symbol, and is where this example connects with the Koblitz problem.

The ring I want to consider is

$$R = \frac{\mathbb{F}_3[\xi_1, \ldots, \xi_{24}]}{\langle ab + cd + ef \rangle}$$

R has no Super DP structure.

Proof:

If R had a Super DP structure, then $\gamma_3(-ab) = a^3\gamma_3(-b) = 0$ should be equal to $\gamma_3(cd + ef)$.

But $\gamma_3(cd + ef) = 2c^2d^2ef + 2cde^2f^2$ by rule 2.

The first way I checked to make sure that this was not zero was using Macaulay2. I checked if $2c^2d^2ef + 2cde^2f^2$ was in the ideal $\langle ab + cd + ef \rangle$. It wasn't.

The Macaulay2 Results

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Macaulay2, version 1.6 with packages: ConwayPolynomials, Elimination, IntegralClosure,
                              LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : p = 3
o1 = 3
i2 : input "super-dp.m2"
ii3 : --This will setup the ring and ideal to do the proper calculations in M2
----I HOPE YOU SET p = A PRIME!
R = ZZ/p [x_1 .. x_(2*(p-1)*6),SkewCommutative=>true]
oo3 = R
oo3 : PolynomialRing
ii4 :        for j from 1 to 6 do y_j = 0
ii5 :        for j from 1 to 6 do
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              for i from (j-1)*(p-1)+1 to (j)*(p-1) do
        y_j = y_j + x_(2*i-1) * x_(2*i)
ii6 : a=y_1
oo6 = x x  + x x
       1 2    3 4
oo6 : R
ii7 : b=y_2
oo7 = x x  + x x
       5 6    7 8
oo7 : R
ii8 : c=y_3
oo8 = x x   + x  x
       9 10    11 12
oo8 : R
ii9 : d=y_4
oo9 = x  x   + x  x
       13 14    15 16
oo9 : R
ii10 : e=y_5
oo10 = x  x   + x  x
        17 18    19 20
oo10 : R
ii11 : f=y_6
oo11 = x  x   + x  x
        21 22    23 24
oo11 : R
ii12 :         I = ideal (a*b+c*d+e*f)
oo12 = ideal(x x x x  + x x x x  + x x x x  + x x x x  + x x  x  x   +
              1 2 5 6    3 4 5 6    1 2 7 8    3 4 7 8    9 10 13 14
-------------------------------------------------------------------------
      x  x  x  x   + x x  x  x   + x  x  x  x   + x  x  x  x   + x  x  x  x   +
       11 12 13 14    9 10 15 16    11 12 15 16    17 18 21 22    19 20 21 22
-------------------------------------------------------------------------
       x  x  x  x   + x  x  x  x  )
        17 18 23 24    19 20 23 24
oo12 : Ideal of R
ii13 :
i14 : 2*c^2*d^2*e*f + 2*c*d*e^2*f^2 % I
o14 = - x x  x  x  x  x  x  x  x  x  x  x   -
         9 10 11 12 13 14 15 16 17 18 21 22
--------------------------------------------------------------------------
      x x  x  x  x  x  x  x  x  x  x  x   - x x  x  x  x  x  x  x  x  x  x  x
       9 10 11 12 13 14 15 16 19 20 21 22    9 10 11 12 13 14 15 16 17 18 23 24
--------------------------------------------------------------------------
      - x x  x  x  x  x  x  x  x  x  x  x   -
         9 10 11 12 13 14 15 16 19 20 23 24
```

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--------------------------------------------------------------------------
      x x  x  x  x  x  x  x  x  x  x  x   - x  x  x  x  x  x  x  x  x  x  x  x
       9 10 13 14 17 18 19 20 21 22 23 24    11 12 13 14 17 18 19 20 21 22 23 24
--------------------------------------------------------------------------
      - x x  x  x  x  x  x  x  x  x  x  x   -
         9 10 15 16 17 18 19 20 21 22 23 24
--------------------------------------------------------------------------
      x  x  x  x  x  x  x  x  x  x  x  x
       11 12 15 16 17 18 19 20 21 22 23 24
o14 : R
i15 : 2*c^2*d^2*e*f + 2*c*d*e^2*f^2 % I == 0
o15 = false
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So in fact $2c^2d^2ef + 2cde^2f^2 \neq 0$.

But the better way (that generalizes without making my computer hang) is by an injection of the "Koblitz ring", $K = \dfrac{\mathbb{F}_3[x_1, \ldots, x_6]}{\langle x_1^3, \ldots, x_6^3, x_1x_2 + x_3x_4 + x_5x_6 \rangle}$. This injection is just a specific example of the more general problem over $\mathbb{F}_p$.